A simpler and better derandomization of an approximation algorithm for Single Source Rent-or-Buy

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Abstract

We present a very simple way of derandomizing the algorithm proposed by Gupta, Kumar and Roughgarden for Single Source Rent-or-Buy by using the method of conditional expectation. Using the improved analysis of Eisenbrand, Grandoni and Rothvoß, our derandomized algorithm has an approximation guarantee of 3.28.

Keywords: Approximation algorithms, Derandomization, Network Design, Rent-or-Buy.

1 Introduction

In the Multicommodity Rent-or-Buy problem, we are given an undirected graph $G = (V, E)$ where each edge $e \in E$ has a nonnegative cost c_e, a set of terminal pairs $\{(s_1, t_1), \ldots, (s_k, t_k)\} \subseteq V \times V$, and a parameter $M > 1$. A solution is a set of paths, one for each vertex pair. For each edge in the solution, we can either choose to *rent* the edge, incurring a cost c_e for each path that uses the edge e, or we can *buy* the edge, incurring a cost Mc_e. The special case when $s_i = s$ for every pair $i = 1, \ldots, k$ is called the Single Source Rent-or-Buy problem.

A related problem is the Connected Facility Location problem. Here we are again given an undirected graph $G = (V, E)$ with nonnegative edge costs c_e, and a parameter $M > 1$, and we are given a set of demand points $D \subseteq V$. A solution is a set of facilities $F \subseteq V$, a connected subgraph T of G that spans F, and an assignment of demand points to open facilities. The cost of a solution is M times the cost of the edges in T, plus for each demand point $j \in D$, the cost of the shortest path $d(i, j)$ from j to the facility $i \in F$ that j is assigned to. Note that we can assume without loss of generality that T is a Steiner tree on F, and that a "root" $r \in V$ is given, such that $r \in F$ must be satisfied for any solution. Single Source Rent-or-Buy can then be formulated as a Connected Facility Location problem with root s and $D = \{t_1, \ldots, t_k\}$, since it is not hard to show that the edges bought in a solution to Single Source Rent-or-Buy form a tree rooted at the source.

Gupta et al. [8] gave the following approximation algorithm SimpleCFL for Connected Facility Location and Single Source Rent-or-Buy, with $q = 1$:

1. Mark each demand $j \in D$ independently with probability $\frac{q}{M}$, and let $\mathbb{D}$ be the *random* set of marked demands.

2. Construct an α-approximate Steiner tree on $\mathbb{D} \cup \{r\}$ and *buy* the edges of this tree.

3. Assign each demand to its closest facility in $\mathbb{D} \cup \{r\}$.

They show that the expected cost of this solution for $q = 1$ is at most $(2 + \alpha)$ times the cost of the optimal solution. Using the current best approximation algorithm [10] for the Steiner tree problem, SimpleCFL is an expected 3.55-approximation algorithm. Eisenbrand et al. [2] recently improved the analysis of this algorithm and showed that the expected cost of the solution is at most $(2+\alpha q)$ times the cost of the optimal solution, if we choose $q = \frac{1}{2} - \frac{1}{\alpha} + \frac{1}{2\alpha}\sqrt{\alpha^2 - 0.772\alpha + 4}$. Using the 1.55-algorithm from [10] in Step 2, this implies that for $q = 0.591$ we get an expected 2.92-approximation algorithm.

A similar type of algorithm was proposed in [6, 7] for Multicommodity Rent-or-Buy. In their *boosted sampling* algorithm for Multicommodity Rent-or-Buy, we mark each terminal pair with probability $\frac{1}{M}$, and buy a Steiner forest $\mathcal{F}$ on the marked pairs. We then connect each remaining terminal pair (s_i, t_i) by renting the edges on the shortest $s_i - t_i$ path in the graph G with the edges of $\mathcal{F}$ contracted. It was shown in [4] that using a primal-dual algorithm [1, 5] to find $\mathcal{F}$, this is an expected 5-approximation algorithm.

It was shown in [9] that a variant of the SimpleCFL algorithm for Single Source Rent-or-Buy yields approximately budget-balanced and group strategyproof cost shares, and that it is possible to derandomize this algorithm and obtain a deterministic 4.2-approximation algorithm for Connected Facility Location and Single Source Rent-or-Buy. No derandomization of the boosted sampling algorithm for Multicommodity Rent-or-Buy is known.

We will show that there is a very simple way of derandomizing SimpleCFL by using the method of conditional expectation [3], with *approximations* of the conditional expected costs, instead of exact expressions. Using a 2-approximation algorithm in Step 2, we obtain a deterministic 4-approximation algorithm for $q = 1$ via the original analysis from [8]. The improved analysis from [2] shows that for $q = 0.636$, our derandomized algorithm has an approximation guarantee of 3.28. Both of these results improve on the previously best known deterministic approximation guarantee of 4.2.

It remains an open question whether it is also possible to use the ideas from this paper to derandomize the boosted sampling algorithm for Multicommodity Rent-or-Buy.

2 Derandomization of SimpleCFL

We start by repeating the lemmas from [8] and [2] from which the expected performance guarantee of SimpleCFL follows. We also repeat the proof of Lemma 2.1, because this will turn out to be helpful in explaining our approximation of the expected buying cost incurred by the algorithm.

In the following, we denote by B^{OPT} and R^{OPT} the buying and renting cost of an optimal solution to the Connected Facility Location problem.

Lemma 2.1 ([8], [2]) *The expected cost of Step 2 of SimpleCFL is at most* $\alpha[B^{OPT} + qR^{OPT}]$.

Proof: We will exhibit a Steiner tree T on $\mathbb{D} \cup \{r\}$ for each realization of $\mathbb{D}$, and we will show that the expected cost of T is at most $B^{OPT} + qR^{OPT}$. It then follows that the expected cost of a minimum-cost Steiner tree is at most $B^{OPT} + qR^{OPT}$.

Let F^* be the set of open facilities in an optimal solution, and let T^* be the Steiner tree on F^* that is bought in the optimal solution. For every $j \in D$, let $i^*(j) \in F^*$ be the facility that j is assigned to in the optimal solution. We define the Steiner tree T on $\mathbb{D} \cup \{r\}$ as the union of the edges of T^* and the edges on the shortest $j - i^*(j)$ path for each $j \in \mathbb{D}$. Since each $j \in D$ is included in $\mathbb{D}$ with probability $\frac{q}{M}$, we buy the edges on the shortest $j - i^*(j)$ path with probability $\frac{q}{M}$, so the expected cost of T is at most $\sum_{e \in T^*} Mc_e + \sum_{j \in D} \frac{q}{M} Md(i^*(j), j) = \sum_{e \in T^*} Mc_e + q\sum_{j \in D} d(i^*(j), j) = B^{OPT} + qR^{OPT}$. □

Lemma 2.2 ([8]) *For $q = 1$, the expected cost of Step 3 of SimpleCFL is at most $2(B^{OPT} + R^{OPT})$.*

Lemma 2.3 ([2]) *The expected cost of Step 3 of SimpleCFL is at most $\frac{0.807}{q}B^{OPT} + 2R^{OPT}$.*

Corollary 2.1 ([8]) *Using $q = 1$, and using the 1.55-approximation algorithm from [10] in Step 2, SimpleCFL is an expected 3.55-approximation algorithm.*

Corollary 2.2 ([2]) *Using $q = 0.591$, and using the 1.55-approximation algorithm from [10] in Step 2, SimpleCFL is an expected 2.92-approximation algorithm.*

In theory, the randomized algorithm SimpleCFL can be derandomized by using the method of conditional expectation [3] as follows: Take an arbitrary vertex j in D. With probability $\frac{q}{M}$, this vertex will be included in $\mathbb{D}$, and with probability $1 - \frac{q}{M}$ it is not included in $\mathbb{D}$. Therefore, the expected cost incurred by SimpleCFL is equal to $\frac{q}{M}$ times the expected cost of SimpleCFL, conditioned on the fact that $j \in \mathbb{D}$ plus $1 - \frac{q}{M}$ times the expected costs of SimpleCFL conditioned on $j \notin \mathbb{D}$. So it must be true that one of the two conditional expected cost is not more than the expected cost of SimpleCFL, and we choose whether or not we will include j in $\mathbb{D}$ to minimize the expected cost conditioned on this choice. We can repeat this process until we have decided for all vertices in D whether or not to include the vertex in $\mathbb{D}$. Since each step does not increase the expected cost of the algorithm, we find a deterministic set $\mathbb{D}$ such that executing Steps 2 and 3 does not cost more than the expected cost of SimpleCFL.

The problem with this approach for this particular problem is that we cannot compute the (conditional) expected cost of SimpleCFL exactly. However, we can compute the conditional expected assignment cost of SimpleCFL, since for each $j \in D$ the expected assignment cost is just the distance to the closest vertex that is included in the random set $\mathbb{D} \cup \{r\}$ (see Claim 2.1).

Furthermore, we will now show that there exists an easily computable value $\bar{c}_{ST}(\mathbb{D} \cup \{r\})$ that is not more than twice the cost of the Steiner tree on $\mathbb{D} \cup \{r\}$ found by the primal-dual algorithm [1, 5]. We will show in the following lemma that given sets $A, \bar{A} \subset D$, such that we have already decided to open facilities in A (so $A \subseteq \mathbb{D}$), and not to open facilities in $\bar{A}$ (so $\bar{A} \cap \mathbb{D} = \emptyset$), that we can compute the conditional expectation of $\bar{c}_{ST}(\mathbb{D} \cup \{r\})$.

Lemma 2.4 *For any subsets $A, \bar{A} \subseteq D$, we can compute $\mathbb{E}[\bar{c}_{ST}(\mathbb{D} \cup \{r\}) | A \subseteq \mathbb{D}, \bar{A} \cap \mathbb{D} = \emptyset]$ in polynomial time such that*

(i) $\mathbb{E}[\bar{c}_{ST}(\mathbb{D} \cup \{r\})] \leq B^{OPT} + qR^{OPT}$;

(ii) The expected cost in Step 2 of SimpleCFL, conditioned on the fact that $A \subseteq \mathbb{D}, \bar{A} \cap \mathbb{D} = \emptyset$, when using a primal-dual algorithm to find a Steiner tree on $\mathbb{D} \cup \{r\}$, is at most $2\mathbb{E}[\bar{c}_{ST}(\mathbb{D} \cup \{r\}) | A \subseteq \mathbb{D}, \bar{A} \cap \mathbb{D} = \emptyset]$.

Proof: We use an idea similar to the one in the proof of Lemma 2.1: In Lemma 2.1 we used an optimal solution to the Connected Facility Location problem to exhibit a Steiner tree on $\mathbb{D} \cup \{r\}$ that costs no more than $B^{OPT} + qR^{OPT}$ in expectation. Here, we will use the optimal solution to an LP relaxation of the Connected Facility Location problem to demonstrate a *fractional* Steiner tree on $\mathbb{D} \cup \{r\}$. The expected cost of this fractional Steiner tree will be at most $B^{OPT} + qR^{OPT}$, and using a primal-dual algorithm we can compute a Steiner tree that costs no more than twice the cost of the fractional Steiner tree.

For a given set $\mathbb{D}$, the cost of buying the Steiner tree found by the primal-dual algorithm on $\mathbb{D} \cup \{r\}$ is at most twice the objective value of any feasible solution to the following LP [1, 5]:

$$
\begin{aligned}
& \min && \sum_{e \in E} M c_e y_e \\
(Buy(\mathbb{D})) \quad & \text{s.t.} && \sum_{e \in \delta(S)} y_e \geq 1 \quad \text{for all } S \text{ s.t. } \mathbb{D} \cap S \neq \emptyset, r \notin S \\
& && y_e \geq 0.
\end{aligned}
$$

We will define a feasible solution to $Buy(\mathbb{D})$ for every possible realization of $\mathbb{D}$ with the help of the following linear programming relaxation for Connected Facility Location [11]. Let x_{ij} be a 0-1 variable that indicates whether demand vertex j is assigned to facility i, and let $z_e = 1$ if edge e is bought, and 0 otherwise. The first set of constraints ensures that every demand is assigned to a facility. For any subset $S \subset V$ that does not contain the root r, such that j is assigned to some facility in S, any feasible solution must buy an edge that leaves S, which is enforced by the second set of constraints.

$$
\begin{aligned}
& \min && B + qR \\
& \text{s.t.} && \sum_{i \in V} x_{ij} = 1 \quad \forall j \in D \\
(CFL) \quad & && \sum_{e \in \delta(S)} z_e \geq \sum_{i \in S} x_{ij} \quad \forall j \in D, S \subset V : r \notin S \\
& && B = \sum_{e \in E} M c_e z_e \\
& && R = \sum_{i \in V} \sum_{j \in D} d(i,j) x_{ij} \\
& && x_{ij}, z_e, B, R \geq 0.
\end{aligned}
$$

Let (x^*, z^*, B^*, R^*) be an optimal solution to CFL. Note that $B^* + qR^* \leq B^{OPT} + qR^{OPT}$, because the optimal solution to the Connected Facility Location problem gives a feasible solution to CFL with objective value $B^{OPT} + qR^{OPT}$.

Let $\chi_e^{ij} = 1$ if e is on the shortest path from demand j to node i (note that χ_e^{ij} is a constant, not a decision variable). For a given set $\mathbb{D}$ let

$$
\tilde{y}_e(\mathbb{D}) = z_e^* + \sum_{j \in \mathbb{D}} \sum_{i \in V} \chi_e^{ij} x_{ij}^* \qquad \forall e \in E.
$$

We define $\bar{c}_{ST}(\mathbb{D} \cup \{r\})$ to be the objective value of $\{\tilde{y}_e(\mathbb{D})\}_{e \in E}$ for $(Buy(\mathbb{D}))$, i.e. $\sum_{e \in E} M c_e \tilde{y}_e(\mathbb{D})$. Note that after solving (CFL) and getting (x^*, z^*) we can compute $E[\bar{c}_{ST}(\mathbb{D} \cup \{r\}) | A \subseteq \mathbb{D}, \bar{A} \cap \mathbb{D} = \emptyset]$ as

$$
E[\bar{c}_{ST}(\mathbb{D} \cup \{r\}) | A \subseteq \mathbb{D}, \bar{A} \cap \mathbb{D} = \emptyset] = \sum_{e \in E} M c_e \left(z_e^* + \sum_{j \in A} \sum_{i \in V} \chi_e^{ij} x_{ij}^* + \sum_{j \in D \setminus (A \cup \bar{A})} \sum_{i \in V} \frac{q}{M} \chi_e^{ij} x_{ij}^* \right).
$$

Then

$$\begin{aligned}
\mathbb{E}[\bar{c}_{ST}(\mathbb{D}\cup\{r\})] &= \sum_{e\in E} Mc_e\Big(z_e^* + \sum_{j\in D}\sum_{i\in V}\frac{q}{M}\chi_e^{ij}x_{ij}^*\Big)\\
&= \sum_{e\in E} Mc_e z_e^* + q\sum_{j\in D}\sum_{i\in V}\Big(\sum_{e\in E}\chi_e^{ij}c_e\Big)x_{ij}^*\\
&= \sum_{e\in E} Mc_e z_e^* + q\sum_{j\in D}\sum_{i\in V} d(i,j)x_{ij}^* = B^* + qR^* \le B^{OPT} + qR^{OPT}.
\end{aligned}$$

It remains to show that $\{\tilde{y}_e(\mathbb{D})\}_{e\in E}$ is a feasible solution to $Buy(\mathbb{D})$: Consider a set S such that $r \notin S$, and there exists $j' \in \mathbb{D}\cap S$. Then

$$\sum_{e\in\delta(S)}\sum_{i\in V}\chi_e^{ij'}x_{ij'}^* \ge \sum_{e\in\delta(S)}\sum_{i\notin S}\chi_e^{ij'}x_{ij'}^* = \sum_{i\notin S}x_{ij'}^*\sum_{e\in\delta(S)}\chi_e^{ij'} \ge \sum_{i\notin S}x_{ij'}^*.$$

The last inequality follows from the fact that if $i \notin S, j' \in S$ then the shortest $i-j'$ path must go from S to $V\backslash S$, so $\sum_{e\in\delta(S)}\chi_e^{ij'} \ge 1$. The feasibility of (x^*, z^*) to (CFL) gives that

$$\sum_{e\in\delta(S)} z_e^* \ge \sum_{i\in S} x_{ij'}^*.$$

Therefore

$$\sum_{e\in\delta(S)}\tilde{y}_e(\mathbb{D}) = \sum_{e\in\delta(S)}(z_e^* + \sum_{j\in\mathbb{D}}\sum_{i\in V}\chi_e^{ij}x_{ij}^*) \ge \sum_{e\in\delta(S)}(z_e^* + \sum_{i\in V}\chi_e^{ij'}x_{ij'}^*) \ge \sum_{i\in S}x_{ij'}^* + \sum_{i\notin S}x_{ij'}^* = 1.$$

So $\{\tilde{y}_e(\mathbb{D})\}_{e\in E}$ is a feasible solution to $Buy(\mathbb{D})$. □

Theorem 2.1 *SimpleCFL can be derandomized to get a deterministic* 4*-approximation algorithm for Connected Facility Location using* $q = 1$*, and a* 3.28*-approximation algorithm using* $q = 0.636$*.*

Proof: We show that for $q \ge 0.636$ we can derandomize SimpleCFL and get a $(2+2q)$-approximation algorithm.

We begin by noting that, even though we are allowed to open a facility in any vertex in V, SimpleCFL only randomly chooses vertices from D to open facilities. In the derandomization, we will therefore also only consider the vertices in D as possible sites for facilities. Suppose the vertices in D are numbered $v_1, \ldots, v_{|D|}$. We will iterate through the vertices in D, and in iteration ℓ we decide whether or not to open a facility in v_ℓ. We let $A \subseteq D$ be the set of vertices for which we have already decided to open a facility, and $\bar{A} \subseteq D$ is the set of vertices for which we have already decided not to open a facility. Initially, $A, \bar{A} = \emptyset$.

Let $\mathbb{E}[c_R(\mathbb{D}\cup\{r\})|A \subseteq \mathbb{D}, \bar{A}\cap\mathbb{D} = \emptyset]$ be the expected assignment cost incurred by SimpleCFL, conditioned on the fact that $A \subseteq \mathbb{D}, \bar{A}\cap\mathbb{D} = \emptyset$.

Claim 2.1 *For given subsets* $A, \bar{A} \subseteq D$*, we can efficiently compute* $\mathbb{E}[c_R(\mathbb{D}\cup\{r\})|A \subseteq \mathbb{D}, \bar{A}\cap\mathbb{D} = \emptyset]$.

For a given demand vertex $j \in D\backslash A$, we order the nodes in $(D\backslash\bar{A})\cup\{r\}$ according to their nondecreasing shortest path distance from j. Let $j(1), j(2), \ldots, j(q)$ be the ordered set of nodes. Let ℓ be the first index such that $j(\ell) \in A\cup\{r\}$, i.e. ℓ is the first index for which we are sure to open a facility at $j(\ell)$, and for all earlier indices k we open a facility at $j(k)$ with probability $\frac{q}{M}$. Then the conditional expected assignment cost for j is

$$\sum_{k=1}^{\ell-1}\left(1-\frac{q}{M}\right)^{k-1}\frac{q}{M}d(j(k),j)+\left(1-\frac{q}{M}\right)^{\ell-1}d(j(\ell),j)$$

and we obtain the total expected assignment cost by summing over all $j \in D\backslash A$. $\diamond$

Let $\mathbb{E}[\bar{c}_{ST}(\mathbb{D}\cup\{r\})|A\subseteq\mathbb{D},\bar{A}\cap\mathbb{D}=\emptyset]$ be given by Lemma 2.4. Let $OPT = B^{OPT}+R^{OPT}$ be the cost of the optimal solution. We maintain the invariant that

$$2\mathbb{E}[\bar{c}_{ST}(\mathbb{D}\cup\{r\})|A\subseteq\mathbb{D},\bar{A}\cap\mathbb{D}=\emptyset]+\mathbb{E}[c_R(\mathbb{D}\cup\{r\}|A\subseteq\mathbb{D},\bar{A}\cap\mathbb{D}=\emptyset]\leq(2+2q)OPT.$$

Note that initially, when $A,\bar{A}=\emptyset$, the invariant holds for $q=1$ by Lemma 2.4(i) and Lemma 2.2. Using the improved analysis from [2], we can make the stronger statement that the invariant initially holds for any $q\geq 0.636$, since by Lemma 2.4(i) and Lemma 2.3 the left hand side is at most $(2+\frac{0.807}{q})B^{OPT}+(2q+2)R^{OPT}$, and $2+\frac{0.807}{q}\leq 2q+2$ for $q\geq 0.636$.

Now, suppose we have already decided for $v_1,\ldots,v_{\ell-1}$ whether or not to open a facility, and $A\subseteq\{v_1,\ldots,v_{\ell-1}\}$ contains the vertices at which we decided to open a facility, and $\bar{A}=\{v_1,\ldots,v_{\ell-1}\}\backslash A$ is the set of vertices at which we will not open a facility.

Since

$$\begin{aligned}&2\mathbb{E}[\bar{c}_{ST}(\mathbb{D}\cup\{r\})|A\subseteq\mathbb{D},\bar{A}\cap\mathbb{D}=\emptyset]+\mathbb{E}[c_R(\mathbb{D}\cup\{r\})|A\subseteq\mathbb{D},\bar{A}\cap\mathbb{D}=\emptyset]=\\ &\frac{q}{M}\Big(2\mathbb{E}[\bar{c}_{ST}(\mathbb{D}\cup\{r\})|A\cup\{v_\ell\}\subseteq\mathbb{D},\bar{A}\cap\mathbb{D}=\emptyset]+\mathbb{E}[c_R(\mathbb{D}\cup\{r\})|A\cup\{v_\ell\}\subseteq\mathbb{D},\bar{A}\cap\mathbb{D}=\emptyset]\Big)\\ &+\left(1-\frac{q}{M}\right)\Big(2\mathbb{E}[\bar{c}_{ST}(\mathbb{D}\cup\{r\})|A\subseteq\mathbb{D},(\bar{A}\cup\{v_\ell\})\cap\mathbb{D}=\emptyset]+\mathbb{E}[c_R(\mathbb{D}\cup\{r\})|A\subseteq\mathbb{D},(\bar{A}\cup\{v_\ell\})\cap\mathbb{D}=\emptyset]\Big),\end{aligned}$$

the smaller of $\Big(2\mathbb{E}[\bar{c}_{ST}(\mathbb{D}\cup\{r\})|A\cup\{v_\ell\}\subseteq\mathbb{D},\bar{A}\cap\mathbb{D}=\emptyset]+\mathbb{E}[c_R(\mathbb{D}\cup\{r\})|A\cup\{v_\ell\}\subseteq\mathbb{D},\bar{A}\cap\mathbb{D}=\emptyset]\Big)$ and $\Big(2\mathbb{E}[\bar{c}_{ST}(\mathbb{D}\cup\{r\})|A\subseteq\mathbb{D},(\bar{A}\cup\{v_\ell\})\cap\mathbb{D}=\emptyset]+\mathbb{E}[c_R(\mathbb{D}\cup\{r\})|A\subseteq\mathbb{D},(\bar{A}\cup\{v_\ell\})\cap\mathbb{D}=\emptyset]\Big)$ must be at most $(2+2q)OPT$ and we choose to include v_ℓ in A or $\bar{A}$ accordingly.

At the end of the last iteration, we have obtained two disjoint sets $A,\bar{A}$ such that $A\cup\bar{A}=D$. Note that the conditional expectations are now constants. By Lemma 2.4(ii), buying the Steiner tree constructed by the primal-dual algorithm on $A\cup\{r\}$ costs at most $2\mathbb{E}[\bar{c}_{ST}(\mathbb{D}\cup\{r\})|A\subseteq\mathbb{D},\bar{A}\cap\mathbb{D}=\emptyset]$, the assignment cost of the vertices in $\bar{A}$ to the closest facility in $A\cup\{r\}$ is equal to $\mathbb{E}[\bar{c}_R(\mathbb{D}\cup\{r\})|A\subseteq\mathbb{D},\bar{A}\cap\mathbb{D}=\emptyset]$, and by the invariant the total cost of this solution is at most $(2+2q)OPT$. $\square$

Remark. The ideas from Lemma 2.4 can also be applied to the boosted sampling algorithm for Multicommodity Rent-or-Buy. However, we do not have an easy way of computing the expected renting costs as we did in Claim 2.1, since the renting cost will depend heavily on the forest that is bought.

Acknowledgement

This research was supported by NSF grant CCF-0514628.

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