The Salesman's Improved Paths

A $\frac{3}{2} + \frac{1}{34}$ approximation

Anke van Zuylen

based on joint work with András Sebő

Workshop on Flexible Network Design, Amsterdam

(Metric) $s - t$ Path TSP

Given:

- vertex set V
- special vertices $s, t \in V$
- distance metric $c : \binom{V}{2} \to \mathbb{Q}^{\geq 0}$

Find:

Minimum length path from s to t that visits every vertex in V.

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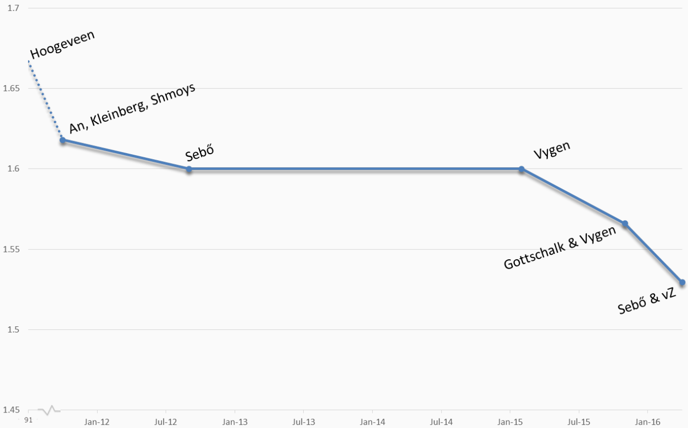
Equivalent to solving $s - t$ path TSP:
Find a minimum cost "$s - t$ tour": a connected multigraph G in which s, t have odd degree and every other vertex has even degree.

$s-t$ Path TSP: Progress for General Metrics



Why does Christofides's algorithm not immediately give a 3/2-approximation?

Christofides for $s - t$ path TSP

Algorithm	Analysis
• Connectivity: find a Minimum Spanning Tree S.	
• Degree parities: add a Min Cost T_S-join J_S:	
$s, t \in T_S$ if they have even degree in S,	
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	Total ???

LP Based Analysis: Subtour Elimination LP for $s-t$ Path TSP

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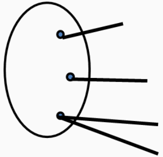

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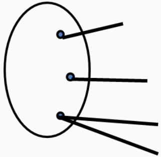

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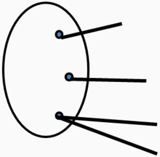

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- At least one edge in an $s-t$ cut
- At least two edges in a non $s-t$ cut

LP Based Analysis I: Connectivity

Subtour LP

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\begin{aligned}
\min \quad & c(x) \\
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Note: Feasible region is a subset of the spanning tree polytope $\Rightarrow$ $c(S) \leq c(x^*)$.

LP Based Analysis: T_S-Join Polyhedron

- $y(e)$ is a decision variable indicating whether e is used in the T_S-join, $y(e) \geq 0$
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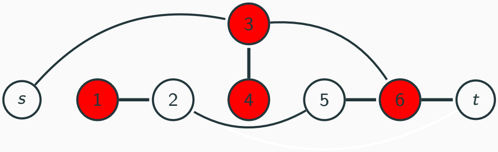

at least one edge in $\delta(U)$ if $U \subset V$ has an odd number of vertices in T_S

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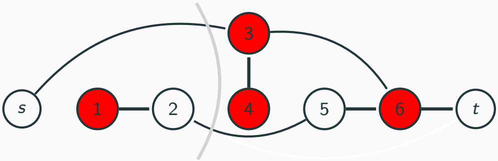

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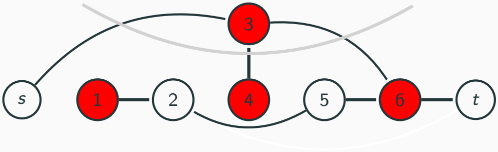

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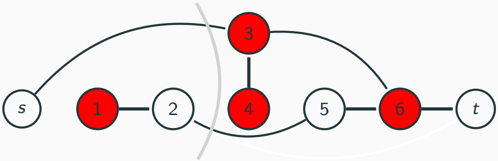

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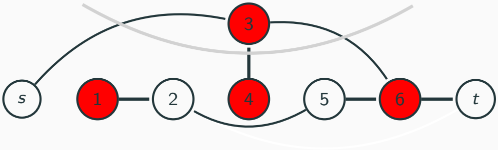

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LP Based Analysis II: Parity

T_S-join polyhedron

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Note: T_S-join polyhedron has integer extreme points $\Rightarrow$ Can upper bound cost of minimum T_S-join by cost of any feasible y-vector.

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Wishful thinking: if $y = x^*/2$ is feasible for the T_S-join polyhedron, then $c(y^*) \leq \frac{1}{2}c(x^*)$.

Conclusion: no $3/2$ yet because...

$x^*/2$ will violate the join constraints for $s-t$ cuts Q such that

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Let the *deficit* for a narrow cut Q be $\left(\frac{2-x^*(Q)}{2}\right)$ "parts-of-an-edge" if $|S \cap Q|$ is even, e.g. an $s-t$ cut Q with $x^*(Q) = \frac{3}{2}$ has deficit $\frac{1}{4}$ if S is even across Q.

How to deal with even narrow cuts?

[AKS'12, S'12, V'14, GV'15]: **Best-of-Many Christofides**

- ✓ Connectivity: Decompose x^* into a collection of spanning trees $\mathcal{S}$ (with multipliers $\lambda_S > 0, \sum_{S\in\mathcal{S}} \lambda_S = 1$).
- ✗ Parity: For each $S \in \mathcal{S}$: add a min cost T_S-join

[Gao'13]: **Gao tree** (for graph $s - t$ path TSP)

- ✗ Connectivity: Compute a minimum cost tree S_{Gao} that has $|S_{Gao} \cap Q| = 1$ *for each narrow cut* Q.
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[This work]: **Best-of-Many with Deletion**

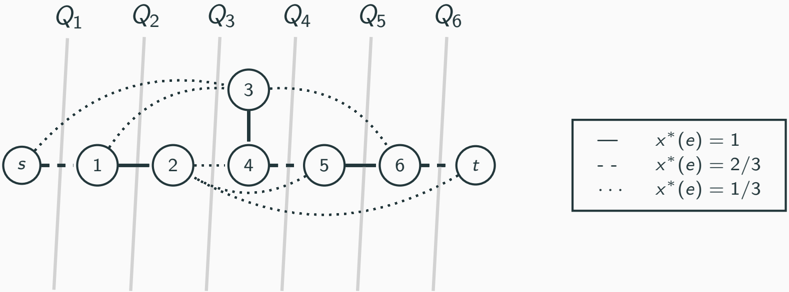
Main idea: For each cut Q, we

- "save" (compared to $c(S) + c(x^*)/2$) on trees S when $|S \cap Q| = 1$
- use the "savings" to make up for Q's deficit in trees S' when $|S' \cap Q|$ is even.

Best-of-Many with Deletion

- Decompose x^* into a collection of spanning trees $\mathcal{S}$ (with multipliers $\lambda_S > 0, \sum_{S \in \mathcal{S}} \lambda_S = 1$) using the Gottschalk-Vygen decomposition.
  - Gottschalk-Vygen guarantees: for narrow cut Q, $|S \cap Q| = 1$ in the "first" $2 - x^*(Q)$ fraction of the trees in the decomposition.
  - In those trees, we will call the edge in $S \cap Q$ "lonely"
- For each $S \in \mathcal{S}$:
  - Delete the "lonely edges"
  - Correct parity (with a join chosen from a specific distribution)
  - Correct connectivity: add back doubled lonely edges if needed.
- Output the solution with minimum cost.

Illustration: LP Solution



Example from Gao'15:

- Narrow cuts are indicated by gray lines and labeled $Q_1, \ldots, Q_6$.
- $x^*(Q_1) = x^*(Q_6) = 1, x^*(Q_i) = \frac{5}{3}$ for $i = 2, 3, 4, 5$.

Illustration: Gottschalk-Vygen Decomposition into Trees

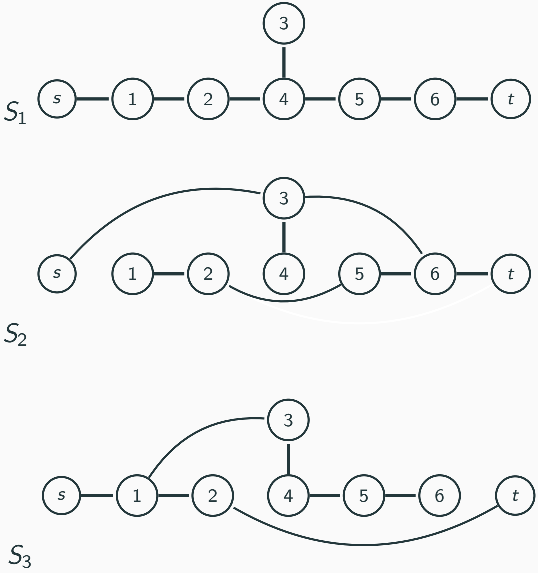
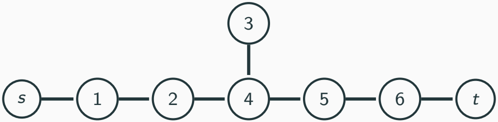


Illustration: Algorithm Executed on Gao-Tree S_1



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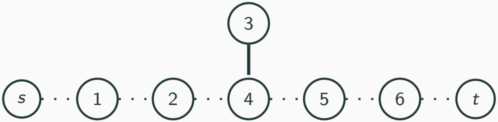

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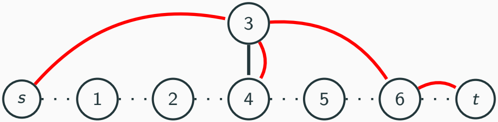

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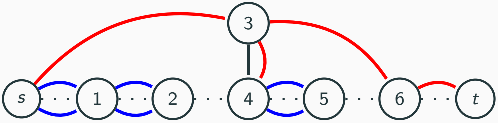

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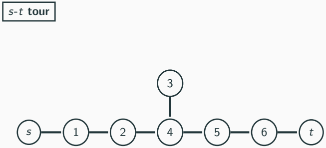
Analysis of Solution Based on S_1



Forest	S_1
Parity Correction	
Connectivity Correction	
Total	

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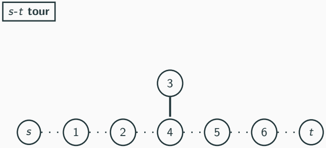

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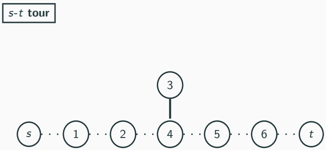
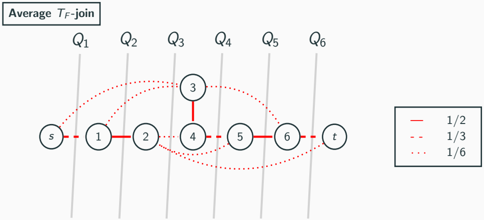
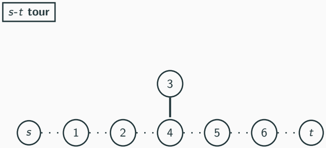
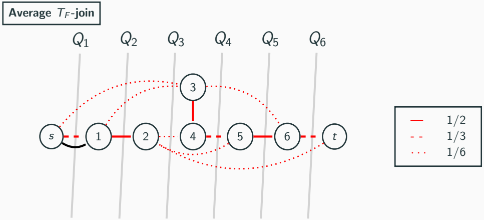




Forest	S_1	$-L_1$
Parity Correction (average)	$\frac{x^*}{2}+$	
Connectivity Correction		
Total		





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Connectivity Correction			
Total			

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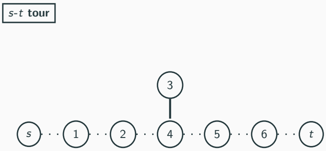
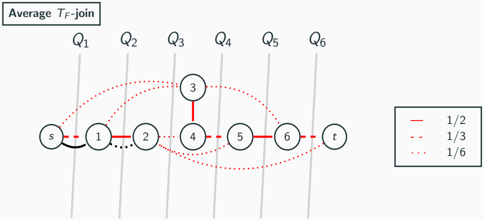
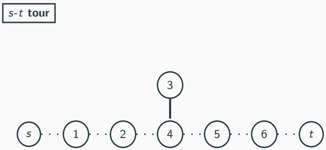
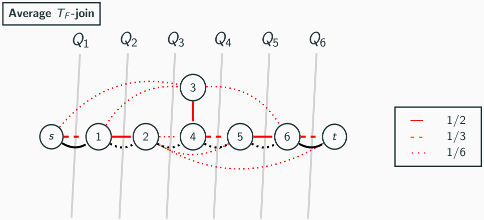




Forest	S_1		$-L_1$
Parity Correction (average)		$\frac{x^*}{2}+$	$\frac{1}{2}(L_1 \cap Q_1)+$ $\frac{1}{6}(L_1 \cap Q_2)+\dots$
Connectivity Correction			
Total			

Analysis of Solution Based on S_1





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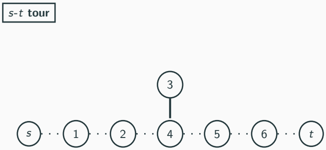

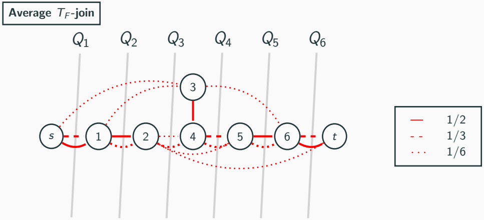

Forest	S_1	$-L_1$
Parity Correction (average)	$\frac{x^*}{2}+$	$\sum_{i=1}^{6} \frac{2-x^*(Q_i)}{2}(L_1 \cap Q_i)$
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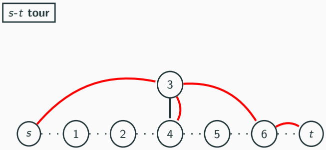

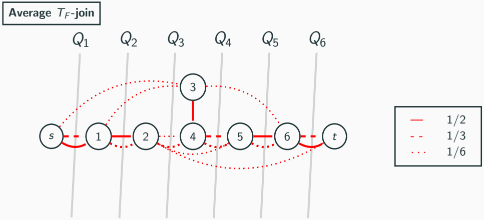

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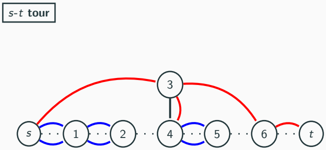

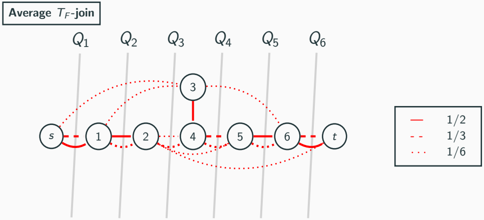

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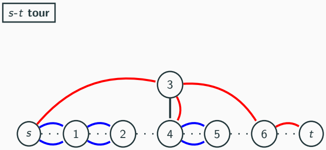

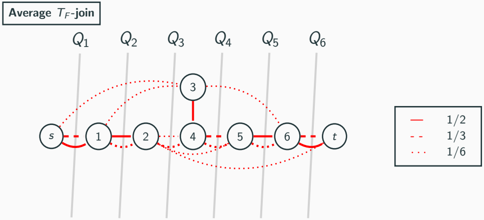

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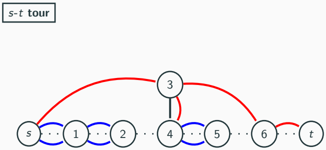

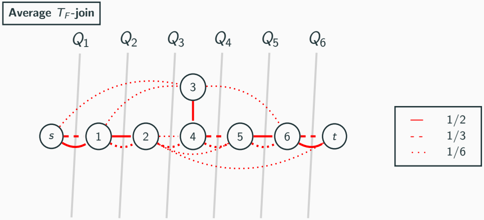

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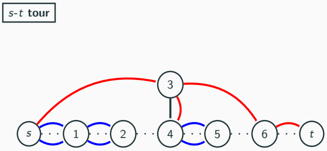

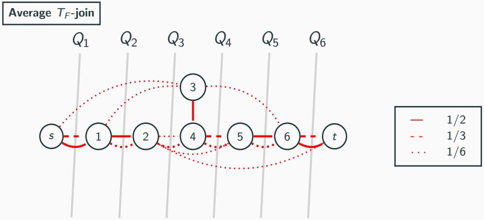

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Connectivity Correction			$\sum_{i=1}^{6}(x^*(Q_i)-1)(L_1 \cap Q_i)$
Total	$S_1+\frac{x^*}{2}$		$-\sum_{i=1}^{6}\frac{2-x^*(Q_i)}{2}(L_1 \cap Q_i)$

How deletion helps

Consider a given narrow cut Q:

- Can show in general:
  If S has a lonely edge in Q, deletion allows us to "save" $\left(\frac{2-x^*(Q)}{2}\right)$-parts-of-an-edge from the total $S + \frac{x^*}{2}$.

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Theorem (SvZ'16)

Let x^* be an optimal solution to the subtour LP, such that no cut Q has $x^*(Q) \in (\frac{3}{2}, 2)$. Then there exists an $s - t$ tour of cost at most $\frac{3}{2}c(x^*)$.

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Can show: If $\gamma = \frac{1}{16}$, then savings can always pay for the deficits.

Improved Result for General Case

Lemma (SvZ'16)
Let x^* be an optimal solution to the subtour LP, and given a decomposition of x^* into spanning trees $\mathcal{S}$, let p^* be the average of the incidence vectors of the $s-t$ path in $\mathcal{S}$. There exists an $s-t$ tour of cost at most $\frac{3}{2}c(x^*) + \frac{1}{16}c(p^*)$.

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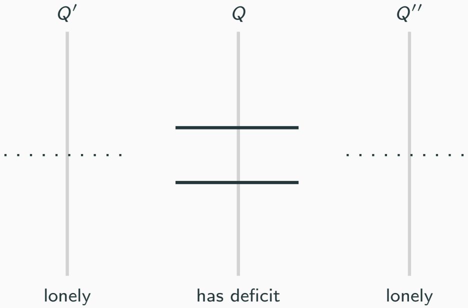
The minimum of the above two tours has cost at most $\left(\frac{3}{2} + \frac{1}{34}\right) c(x^*)$.

What I did not show you (but did use):

how to prove that Connectivity Correction can be done using at most

$$\sum_{Q \text{ lonely in S}} (x^*(Q) - 1)(L \cap Q)$$

A few details on the proof: need for Gottschalk-Vygen



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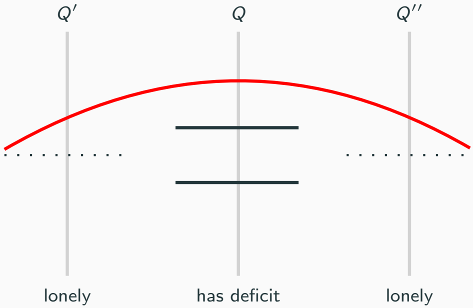

red edge is bad
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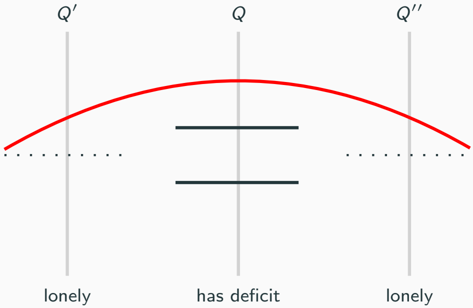

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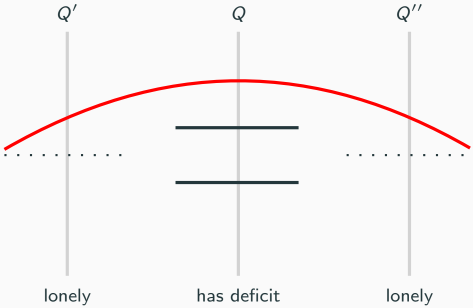

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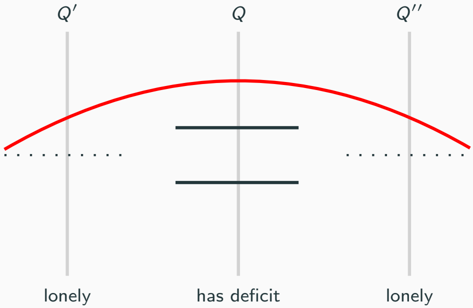

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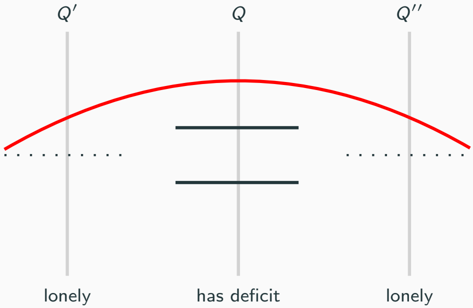

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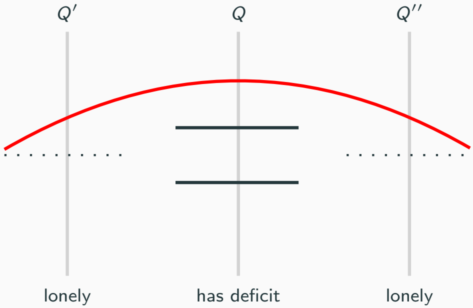

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$\Rightarrow$ Q's saved edge-parts are not bad.

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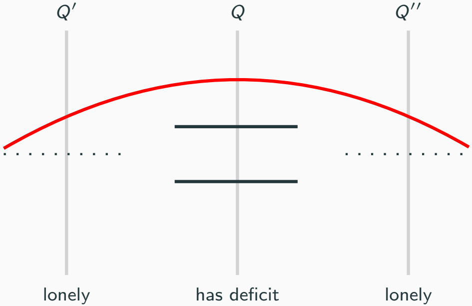

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For each bad edge (that crosses more than one cut with a lonely edge), we double all but one of the lonely edges to reconnect.

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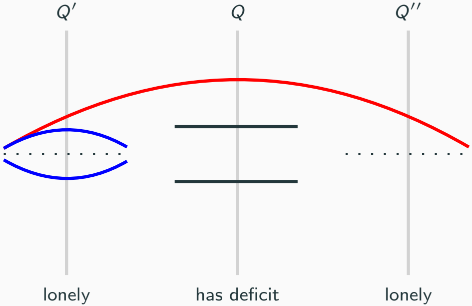

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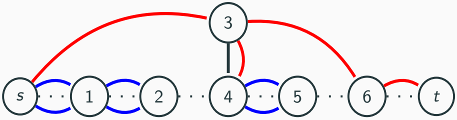
The solution to a linear program is used to decide *which* lonely edges are used for reconnection (only needed to "spread things out" for the analysis: the algorithm may use the cheapest lonely edges).

Conclusion



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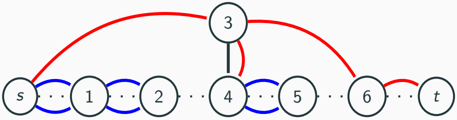

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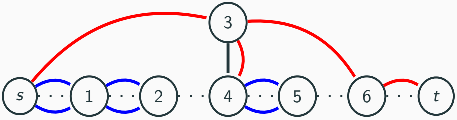

- Approximation guarantee and integrality gap bound of $\frac{3}{2} + \frac{1}{34}$
- Mixing Parity and Connectivity stages in new ways leads to benefits!
- Open question: Shave off the remaining $+\frac{1}{34}$ from the ratio.

Questions??